

Parity Biquandle Invariants of Virtual Knots

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Abstract

We define counting and cocycle enhancement invariants of virtual knots using parity biquandles. The cocycle invariants are determined by pairs consisting of a biquandle 2-cocycle ϕ^0 and a map ϕ^1 with certain compatibility conditions leading to one-variable or two-variable polynomial invariants of virtual knots. We provide examples to show that the parity cocycle invariants can distinguish virtual knots which are not distinguished by the corresponding non-parity invariants.

KEYWORDS: Virtual knots, parity biquandles, cocycle invariants, enhancements of counting invariants

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1 Introduction

In 1996, Louis Kauffman introduced the world to *virtual knot theory* in [11]. Virtual knot theory is a combinatorial generalization of the theory of knotted curves in \mathbb{R}^3 , now known as *classical knot theory*. Each ambient isotopy class of knotted oriented curves in \mathbb{R}^3 coincides with an equivalence class of combinatorial objects known as *signed Gauss diagrams*; however, the set of all such equivalence classes includes classes which do not correspond to classical knots. These extra classes are known as *virtual knots*. Virtual knots can be understood geometrically as knots in certain 3-manifolds ($\Sigma \times [0, 1]$ for Σ an orientable surface) up to equivalence by stabilization of Σ [4].

In [11], it is observed that every classical knot is represented by a diagram in which every crossing is *evenly intersticed*, i.e. every crossing has an even number of over- and under-crossing points along the knot between its over and under instances. In virtual knots, a crossing can have an even or odd number of crossing points between its over and under instances, and moreover this even or odd *parity* is not changed by Reidemeister moves. In [13], parity was used to create a number of invariants for virtual knots. In [7] the notion of parity was generalized to integer-valued maps and used to define new invariants of virtual knots. Parity invariants are very good at distinguishing classical knots from non-classical virtual knots as well as simply distinguishing virtual knot types.

In [8] (and see also [12]), algebraic structures known as *biquandles* were introduced. Given any finite biquandle X , there is a non-negative integer-valued invariant of oriented knots and links known as the *biquandle counting invariant* which counts homomorphisms from the fundamental biquandle of a knot K to X , represented as colorings of the semiarcs of K by elements of X . Cocycles in the second cohomology of a finite biquandle were first used to enhance the biquandle counting invariant in [3].

In [10], biquandles incorporating the notion of parity were introduced (see also [1]). In this paper we extend the counting invariant to the case of finite parity biquandles and define enhancements of the counting invariant using *parity enhanced cocycles*, cocycles in the second cohomology of the even part of the parity biquandle with extra information analogous to the virtual cocycles in [6].

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The paper is organized as follows. In Section 2 we review the basics of biquandles. In Section 3 we review parity biquandles and introduce the parity biquandle counting invariant. In Section 4 we review biquandle cohomology and define parity cocycle enhancements of the counting invariant. We provide examples demonstrating that the parity enhanced cocycle invariants are stronger than the corresponding unenhanced cocycle invariants and the corresponding non-parity invariants for virtual knots. In Section 5 we conclude with questions for future research.

2 Biquandles

A biquandle is an algebraic structure with axioms motivated by the Reidemeister moves (see [8, 12] etc.). It can be defined abstractly:

Definition 1. A *biquandle* is a set X along with two operators, $\bar{\triangleright}$ and \triangleright , both maps $X \times X \mapsto X \times X$, such that:

- (i) For all $x \in X$, $x \bar{\triangleright} x = x \triangleright x$
- (ii) We have right invertibility of both maps and pairwise invertibility, i.e. the maps $\alpha_y : x \mapsto x \bar{\triangleright} y$, $\beta_y : x \mapsto x \triangleright y$, and $S : (x, y) \mapsto (y \bar{\triangleright} x, x \triangleright y)$ are all invertible.
- (iii) For all $x, y, z \in X$, we have the *exchange laws*:

$$\begin{aligned} (z \bar{\triangleright} y) \bar{\triangleright} (x \bar{\triangleright} y) &= (z \bar{\triangleright} x) \bar{\triangleright} (y \triangleright x) \\ (x \bar{\triangleright} y) \triangleright (z \bar{\triangleright} y) &= (x \triangleright z) \bar{\triangleright} (y \triangleright z) \\ (y \triangleright x) \triangleright (z \bar{\triangleright} x) &= (y \triangleright z) \triangleright (x \triangleright z) \end{aligned}$$

Example 1. A well-known type of biquandle is the *Alexander biquandle*. The biquandle's underlying set X is a module over the ring $\Lambda = \mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$ of two-variable Laurent polynomials. In particular, note that s and t are invertible, so for Alexander biquandles structures on finite rings or fields (where s and t are elements of the ring), the characteristic must be relatively prime to s and t . The operations are defined as:

$$\begin{aligned} x \triangleright y &= tx + (s^{-1} - t)y \\ x \bar{\triangleright} y &= s^{-1}y \end{aligned}$$

The first biquandle axiom follows from the definition, the second follows from the fact that s and t are invertible, and the exchange laws can be easily checked.

Example 2. Given a finite set $X = \{x_1, \dots, x_n\}$, we can define biquandle structures on X by encoding the operation tables of \triangleright and $\bar{\triangleright}$ as blocks in a matrix. For example, the set $X = \{x_1, x_2, x_3\}$ is a biquandle with operations defined by the operation tables

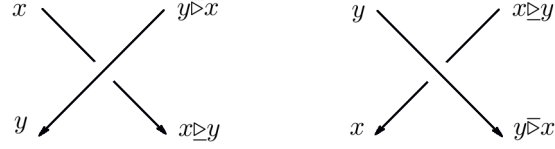
$$\begin{array}{c|ccc} \triangleright & x_1 & x_2 & x_3 \\ \hline x_1 & x_1 & x_3 & x_2 \\ x_2 & x_3 & x_2 & x_1 \\ x_3 & x_2 & x_1 & x_3 \end{array} \quad \text{and} \quad \begin{array}{c|ccc} \bar{\triangleright} & x_1 & x_2 & x_3 \\ \hline x_1 & x_1 & x_1 & x_1 \\ x_2 & x_2 & x_2 & x_2 \\ x_3 & x_3 & x_3 & x_3 \end{array}$$

which we abbreviate by dropping the “ x ”s to obtain the biquandle matrix

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 & 2 & 2 \\ 2 & 1 & 3 & 3 & 3 & 3 \end{array} \right].$$

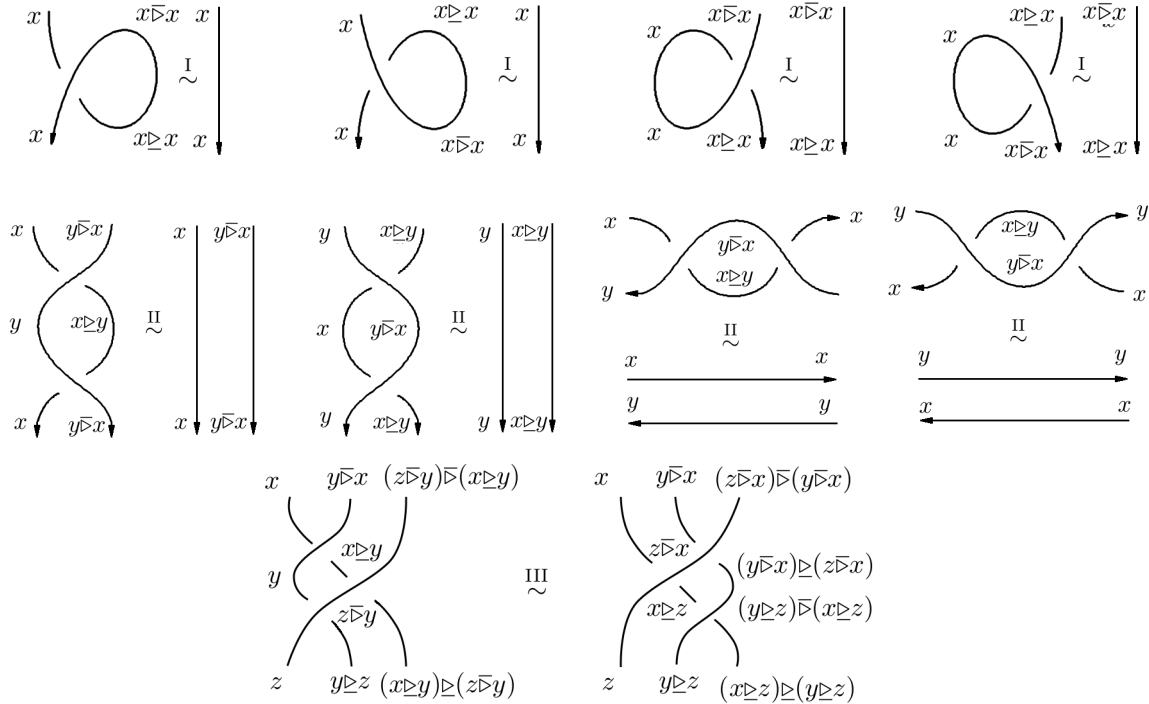
The left-hand block represents the operation \triangleright while the right-hand block represents the operation $\bar{\triangleright}$; then for instance we have $x_2 \triangleright x_3 = x_1$ and $x_3 \bar{\triangleright} x_1 = x_3$, obtained by looking up the entries in row 2 column 3 and row 3 column 1 of the left and right blocks respectively.

To construct knot invariants using finite biquandles, we assign a biquandle element to each semi-arc of a knot such that the pictured *coloring condition*:



is satisfied at every crossing. This is sometimes called “coloring” the knot, and can be understood as a homomorphism from the *fundamental biquandle* of the knot to the coloring biquandle. (The fundamental biquandle of a knot K , denoted $\mathcal{B}(K)$, is the set of equivalence classes of biquandle words generated by semiarcs in a diagram of K modulo the crossing relations and biquandle operations). As in the picture, if a crossing is oriented downward as shown, the colorings of the two left-hand semi-arcs together with the two operators determine the colorings of the two right-hand semi-arcs. In particular, we may interpret $x \supseteq y$ as x after going under y , and $x \bar{\supseteq} y$ as x after going over y . Given a particular coloring biquandle and a particular knot, each crossing yields a constraint on the possible colorings of the knot with that biquandle.

Indeed, the biquandle axioms are chosen precisely to guarantee that for any biquandle coloring of a knot or link diagram before a Reidemeister move, there is a unique corresponding coloring after the move:



Axiom 1 ensures that the semiarc created by a Reidemeister I move has a well-defined coloring, and Axiom 2 ensures a 1-1 correspondence between colorings before and after the move. The three invertibility conditions of Axiom 2 ensure 1-1 correspondences in different orientations of Reidemeister II moves. The exchange laws from Axiom 3 follow from the boundary conditions and multiple colorings in a Reidemeister III move. Note that Reidemeister I and II moves allow us to move between all the forms of the Reidemeister III move, permitting us to consider only the case with all three crossings positive.

Note that it is common elsewhere in the literature to define biquandle operations with the inbound oriented semiarcs operating on each other to produce the outbound oriented semiarcs, which we might

call “top-down” operations; however, the “sideways” operations are historically first (see [8]) and have the advantages of resulting in more symmetric axioms and making biquandle homology much simpler.

We want to use biquandles to define an invariant based on assigning elements of a biquandle to knot semi-arcs. Given a particular biquandle X , we have a counting invariant, which is the number of ways of assigning biquandle elements to a given knot.

Definition 2. Let X be a finite biquandle and K an oriented knot. The *biquandle counting invariant* is the number of biquandle colorings of K by X ,

$$\Phi_X^{\mathbb{Z}}(L) = |\text{Hom}(\mathcal{B}(K), X)|.$$

As outlined above, every valid biquandle labeling of a knot diagram before a Reidemeister move corresponds to a unique valid biquandle labeling of the diagram after the move. This ensures that the above is indeed a knot invariant.

Theorem 1. *For any finite biquandle X , the corresponding biquandle counting invariant is knot invariant.*

3 Parity Biquandles

To refine the biquandle counting invariant, we use an extension of the biquandle, the *parity biquandle*. First defined in [10], parity biquandles are similar to biquandles, but with four operations instead of just two: $\bar{\triangleright}^0, \bar{\triangleright}^1, \underline{\triangleright}^0, \underline{\triangleright}^1$, and some additional restrictions.

Definition 3. A *parity biquandle* is a set X along with four operations: $\bar{\triangleright}^0, \bar{\triangleright}^1, \underline{\triangleright}^0, \underline{\triangleright}^1$, all maps $X \times X \mapsto X \times X$, such that:

- (i) X along with the two operations $\bar{\triangleright}^0$ and $\underline{\triangleright}^0$ is a biquandle (X along with $\bar{\triangleright}^1$ and $\underline{\triangleright}^1$ need not be).
- (ii) We have right invertibility of both odd maps and pairwise invertibility of those maps, i.e. the maps $\alpha_y^1 : x \mapsto x \bar{\triangleright}^1 y, \beta_y^1 : x \mapsto x \underline{\triangleright}^1 y$, and $S : (x, y) \mapsto (y \bar{\triangleright}^1 x, x \underline{\triangleright}^1 y)$ are all invertible.
- (iii) We have the *mixed exchange laws*:

$$\begin{aligned} (z \bar{\triangleright}^a y) \bar{\triangleright}^b (x \bar{\triangleright}^c y) &= (z \bar{\triangleright}^b x) \bar{\triangleright}^a (y \underline{\triangleright}^c x) \\ (x \bar{\triangleright}^a y) \underline{\triangleright}^b (z \bar{\triangleright}^c y) &= (x \underline{\triangleright}^b z) \bar{\triangleright}^a (y \underline{\triangleright}^c z) \\ (y \underline{\triangleright}^a x) \underline{\triangleright}^b (z \bar{\triangleright}^c x) &= (y \underline{\triangleright}^b z) \underline{\triangleright}^a (x \underline{\triangleright}^c z) \end{aligned}$$

for $(a, b, c) \in \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ (The case where $a = b = c = 0$ must hold, but is enforced by condition 1).

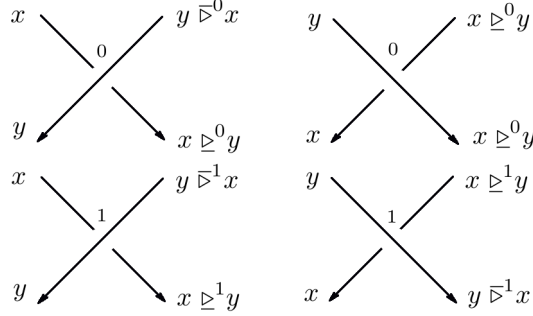
Note that biquandles are the special case of parity biquandles where $x \underline{\triangleright}^1 y = x \underline{\triangleright}^0 y$, and $x \bar{\triangleright}^1 y = x \bar{\triangleright}^0 y$.

For coloring virtual knots with parity biquandles, we will use the following definition (see also [10]):

Definition 4. Let K be a virtual knot diagram and let C be a classical crossing in K . We will say C has *parity 0* if the number of classical over and under crossings encountered traveling along K between the under and over instances of C is even, and we will say C has *parity 1* if the number of classical over and under crossings encountered traveling along K between the under and over instances of C is odd.

Parity biquandles can be used to capture additional structure in virtual knots by using operators based on the parity of each crossing, with the 1 superscript for odd crossings and the 0 superscript for even ones. We will occasionally find it convenient to decorate each crossing with a 0 or a 1 to explicitly indicate its parity.

Given a parity biquandle X , we color the semiarcs of a virtual knot diagram D with elements of X , with constraints generated by the even operations \succeq^0, \preceq^0 at even crossings and the odd operations \succeq^1, \preceq^1 at the odd crossings.



Remark 1. Note that parity biquandle colorings only differ from biquandle colorings in non-classical virtual knots, since classical knots have only even crossings. This makes the odd operators irrelevant for classical knots.

As with a biquandle, given a parity biquandle X , we can find the number of ways of assigning elements of X to semi-arcs of a knot K , or “coloring” K , respecting the parity biquandle’s relations. As before, these colorings can be defined as homomorphisms from $\mathcal{PB}(K)$, the *fundamental parity biquandle* of K , to the coloring biquandle X . (The fundamental parity biquandle is the set of equivalence classes of parity biquandle words generated by semiarcs in a diagram of K modulo the crossing relations and parity biquandle operations.)

Definition 5. The *parity biquandle counting invariant* is defined by

$$\Phi_X^{\mathbb{Z}}(K) = |\text{Hom}(\mathcal{PB}(K), X)|$$

To show that this is indeed an invariant, we must show that a single coloring before a Reidemeister move implies a unique coloring after the move. To do this, we look at the moves one at a time:

For type I moves, the crossing involved in the move is always even, so the biquandle condition on the even-crossing operators forces invariance.

For type II moves, we note that the signs of both crossings must be the same (by inspection of Gauss diagram). If both are even, invariance is forced by the biquandle condition on the even-crossing operators. If both are odd, invariance is forced by the invertibility rules from the definition.

For type III moves, the biquandle condition forces the all-even exchange laws. This forces invariance under the all-even Reidemeister III move. To show that the mixed exchange laws capture all the other cases, we need a lemma:

Lemma 2. *In a Reidemeister III move, either all three crossings involved are even, or two are odd and one is even.*

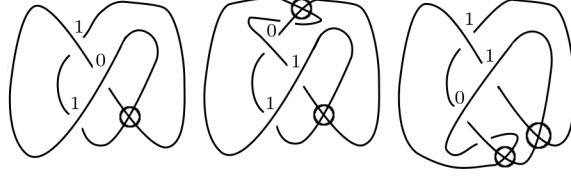
Proof. Consider the possible Gauss diagrams that might start a type III move. Below we have diagrams of all the possible starting positions for the crossings involved in the move. (For more about Gauss diagrams, see for instance [9]). We can divide the crossing labels not involved in the move into three sections (i.e., the dotted portions of the outer circle), based on their “minor segment” in the diagrams below.



This total will be even, so either all three partitions are even or two are odd and one is even. Each minor segment's parity determines the parity of the crossing corresponding with that segment's chord. So the crossing parities must follow the same pattern: either all are even, or one is even and two odd. \square

Given Lemma 2, it is clear that the mixed exchange laws encompass all of the remaining cases.

We can construct examples to show that all of the cases not ruled out by Lemma 2 are in fact possible. This is why all three cases of the mixed exchange laws are necessary. The figure below shows three knots where a Reidemeister III move is possible, each with the relevant crossing parities labeled.



We now have our desired result.

Theorem 3. *For any finite parity biquandle X , the corresponding parity biquandle counting invariant is a knot invariant.*

Example 3. As an example of a parity biquandle structure, we construct a family of parity biquandles by extending the Alexander biquandle structure. An *Alexander parity biquandle* is a module X over the ring $\Lambda = \mathbb{Z}[t^{\pm 1}, s^{\pm 1}, b^{\pm 1}, a^{\pm 1}]$ of four-variable Laurent polynomials (i.e. with all four variables invertible in the coefficient ring). The following constraints must also be satisfied:

$$\begin{aligned} (a^{-1} - b)^2 + (s^{-1} - t)(b - a^{-1}) &= 0 \\ (a^{-1} - b)(b - t) &= 0 \\ (a^{-1} - b)(s^{-1} - a^{-1}) &= 0 \end{aligned}$$

The operators are defined as:

$$\begin{aligned} x \rhd^0 y &= tx + (s^{-1} - t)y \\ x \rhd^1 y &= s^{-1}y \\ x \lhd^1 y &= bx + (a^{-1} - b)y \\ x \lhd^0 y &= a^{-1}y \end{aligned}$$

The first two parity biquandle axioms follow from the definition and the (mixed) exchange laws follow from the constraints.

Example 4. Given a finite set $X = \{x_1, \dots, x_n\}$ we can define parity biquandle structures on X with a $2n \times 2n$ block matrix M encoding the operation tables of the even and odd operations such that $x_i \rhd^\epsilon x_j = M_{i,j+\epsilon n}$ and $x_i \lhd^\epsilon x_j = M_{i+n,j+\epsilon n}$, where $\epsilon \in \{0, 1\}$ and $M_{i,j}$ is the entry of M in row i column j . That is, we will encode the operations tables as a block matrix whose blocks are the operation tables of the operations arranged as $\begin{bmatrix} \rhd^0 & \rhd^1 \\ \lhd^0 & \lhd^1 \end{bmatrix}$. For example, the set $X = \{x_1, x_2, x_3\}$ has parity biquandle structures including

$$\begin{bmatrix} \begin{array}{ccc|ccc} 3 & 1 & 3 & 3 & 1 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 3 & 1 & 1 & 3 & 1 \\ \hline 1 & 3 & 1 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 1 & 1 & 1 \end{array} \end{bmatrix}.$$

Then in this case, we have $3 \rhd^0 1 = 1$ and $1 \lhd^1 2 = 3$.

4 Parity Cocycle Enhancements

We begin this section with a brief review of biquandle homology; see [3, 5, 6], etc., for more.

Let X be a finite biquandle and A an abelian group. Define $C_n(X; A) = A[X^n]$, the free A -module on ordered n -tuples of elements of X . For each $n = 1, 2, 3, \dots$, define $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ by setting

$$\partial_n(x_1, \dots, x_n) = \sum_{k=1}^n (-1)^k (\partial_{n,k}^0(x_1, \dots, x_n) - \partial_{n,k}^1(x_1, \dots, x_n))$$

where

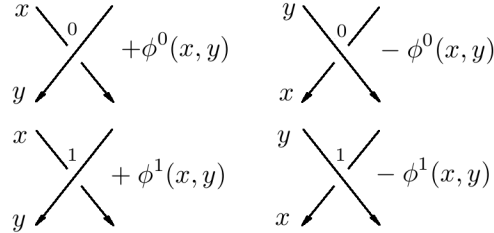
$$\begin{aligned} \partial_{n,k}^0(x_1, \dots, x_n) &= (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \\ \partial_{n,k}^1(x_1, \dots, x_n) &= (x_1 \triangleright x_k, \dots, x_{k-1} \triangleright x_k, x_{k+1} \bar{\triangleright} x_k, \dots, x_n \bar{\triangleright} x_k) \end{aligned}$$

and extending linearly.

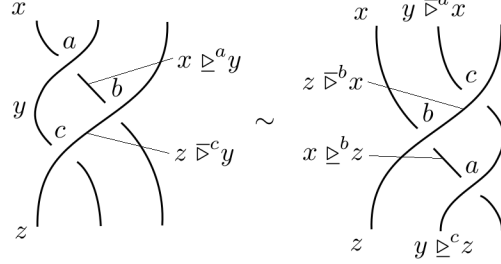
Then (see [5] for example) ∂ is a boundary map; the A -modules $H_n(X; A) = \text{Ker}(\partial_n)/\text{Im}(\partial_{n+1})$ and $H^n(X; A) = \text{Ker}(d^{n+1})/\text{Im}(d^n)$ (where $d^n(f(x)) = f(\partial_n(x))$ for any $f : C_n(X) \rightarrow A$, i.e. for any $f \in C^n$) are the n th birack homology and cohomology modules with coefficients in A respectively. (A birack is like a biquandle but without the conditions resulting from the type I move replaced with conditions arising from the framed type I move; see for instance [14] for more). A birack cocycle which evaluates to zero on all degenerate chains (A -linear combinations of generators (x_1, \dots, x_n) with $x_k = x_{k+1}$ for some $k = 1, \dots, n-1$) is a reduced cocycle.

In [3, 6] and more, biquandle 2-cocycles are used to define enhancements of the biquandle counting invariant for finite biquandles. Specifically, for any X -colored knot or link diagram, a reduced 2-cocycle ϕ is evaluated on the colors at each crossing; the algebraic sum of these cocycle values (i.e. $+\phi(x, y)$ at positive crossings and $-\phi(x, y)$ at negative crossings, where x, y are the under- and over-crossing colors at the crossing), known as a *Boltzmann weight*, is then invariant under X -labeled Reidemeister moves. Then the multiset of Boltzmann weights over the set of X -colorings of L is an enhanced invariant of L with cardinality equal to the X -counting invariant. It is common to rewrite this multiset as a polynomial by taking the generating function of the multiset, i.e. converting multiplicities to integer coefficients and multiset elements to exponents of a dummy variable u .

To incorporate parity, we define the notion of a *parity enhanced biquandle 2-cocycle*, which is a reduced 2-cocycle $\phi^0 \in C^2(X; A)$ paired with another function $\phi^1 : X \times X \rightarrow A$ satisfying certain compatibility conditions with ϕ^0 . Essentially, the idea is to evaluate ϕ^0 at even crossings and ϕ^1 at odd crossings.



Looking at the Reidemeister III move, we have



and hence we need

$$\phi^a(x, y) + \phi^b(x \succeq^a y, x \bar{\succeq}^c y) + \phi^c(y, z) = \phi^a(x \succeq^b z, y \succeq^c z) + \phi^b(x, z) + \phi^c(y \bar{\succeq}^a x, z \bar{\succeq}^b x)$$

for triples $(a, b, c) \in \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$.

Analogously to [6], we can define two forms of compatibility between ϕ^0 and ϕ^1 :

Definition 6. Let X be a finite biquandle, A an abelian group and $\phi^0, \phi^1 : A[X^2] \rightarrow A$ linear maps. We say ϕ^0 and ϕ^1 are *compatible* if for all $x, y, z \in X$ and for all $(a, b, c) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ we have

$$\phi^a(x, y) + \phi^b(x \succeq^a y, x \bar{\succeq}^c y) + \phi^c(y, z) = \phi^a(x \succeq^b z, y \succeq^c z) + \phi^b(x, z) + \phi^c(y \bar{\succeq}^a x, z \bar{\succeq}^b x).$$

We say ϕ^0 and ϕ^1 are *strongly compatible* if ϕ^0 and ϕ^1 are compatible and we additionally have

$$\begin{aligned} \phi^0(y, z) &= \phi^0(y \bar{\succeq}^1 x, z \bar{\succeq}^1 x), \\ \phi^0(x, z) &= \phi^0(x \succeq^1 y, z \bar{\succeq}^1 y), \text{ and} \\ \phi^0(x, y) &= \phi^0(x \succeq^1 z, y \succeq^1 z) \end{aligned}$$

for all $x, y, z \in X$.

The compatibility condition together with the 2-cocycle condition for ϕ^0 guarantees that the sum $\phi^j(x, y)$ of contributions from ϕ^0 and ϕ^1 at even and odd crossings is not changed by Reidemeister III moves. The strong compatibility condition guarantees that the separate contributions from even and odd crossings are preserved by type III moves. The contribution rules guarantee that (even or odd) type II moves do not change the total contribution, and the reduced condition for ϕ^0 guarantees that type I moves do not change the overall sum of crossing weights. Thus, we have:

Definition 7. Let X be a finite parity biquandle, K a virtual knot, A an abelian group and $\phi^0 \in H^2(X; A)$ a reduced biquandle 2-cocycle, and $\phi^1 : A[X^2] \rightarrow A$ a linear map compatible with ϕ^0 . For each $f \in \text{Hom}(\mathcal{PB}(K), X)$, the *parity Boltzmann weight* of f is the sum

$$BW(f) = \sum_{c \text{ crossings}} \sigma(c) \phi^{\epsilon(c)}(x_c, y_c)$$

where $\sigma(c) = \pm 1$ is the sign of the crossing, $\epsilon(c) \in \{0, 1\}$ is the parity of the crossing, and (x_c, y_c) are the left side under- and over-crossing labels. If ϕ^0 and ϕ^1 are strongly compatible, the *strong parity Boltzmann weight* of f is

$$SBW(f) = (SBW(f)_0, SBW(f)_1) = \left(\sum_{\text{even crossings}} \sigma(c) \phi^0(x_c, y_c), \sum_{\text{odd crossings}} \sigma(c) \phi^1(x_c, y_c) \right).$$

Then the *parity enhanced biquandle cocycle multiset* of K is the multiset

$$\Phi_X^{\phi,M}(K) = \{BW(f) \mid f \in \text{Hom}(\mathcal{PB}(K), X)\}$$

or

$$\Phi_X^{\phi,sM}(K) = \{SBW(f) \mid f \in \text{Hom}(\mathcal{PB}(K), X)\}$$

in the strongly compatible case. The *parity enhanced biquandle cocycle polynomial* of K is

$$\Phi_X^\phi(K) = \sum_{f \in \text{Hom}(\mathcal{PB}(K), X)} u^{BW(f)}$$

or

$$\Phi_X^{\phi,s}(K) = \sum_{f \in \text{Hom}(\mathcal{PB}(K), X)} u^{SBW(f)_0} v^{SBW(f)_1}$$

in the strongly compatible case.

By construction, we have our main result:

Proposition 4. *Let X be a parity biquandle and ϕ a parity-enhanced cocycle. If two virtual knots K and K' are related by virtual Reidemeister moves, then*

$$\Phi_X^{\phi,M}(K) = \Phi_X^{\phi,M}(K') \quad \text{and} \quad \Phi_X^\phi(K) = \Phi_X^\phi(K').$$

If ϕ^0 and ϕ^1 are strongly compatible, we have

$$\Phi_X^{\phi,sM}(K) = \Phi_X^{\phi,sM}(K') \quad \text{and} \quad \Phi_X^{\phi,s}(K) = \Phi_X^{\phi,s}(K').$$

Therefore, all four are knot invariants.

We can conveniently specify any pair $\phi^0, \phi^1 : X \times X \rightarrow A$ with an $n \times 2n$ block matrix with entries in A representing the coefficients of the characteristic maps χ_{x_i, x_j} . For instance, if $X = \{x_1, x_2\}$ then we use the matrix

$$\left[\begin{array}{cc|cc} 2 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array} \right]$$

to indicate the maps $\phi^0 = 2\chi_{(x_1, x_1)} + \chi_{x_1, x_2} - \chi_{(x_2, x_2)}$ and $\phi^1 = \chi_{(x_1, x_2)} + \chi_{x_2, x_1} - 2\chi_{(x_2, x_2)}$.

Remark 2. If ϕ^0 is a biquandle 2-cocycle and ϕ^1 and ψ^1 are both strongly compatible with ϕ^0 , then we note that $\phi^1 + \psi^1$ and $\alpha\phi^1$ for $\alpha \in A$ are also strongly compatible with ϕ^0 . In particular, for each biquandle 2-cocycle, the set of strongly compatible maps has the structure of an A -module.

Our first example illustrates the computation of the invariant and demonstrates that the parity cocycle enhancement provides more information than the corresponding unenhanced biquandle counting invariant.

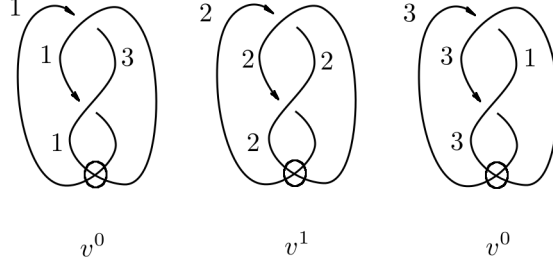
Example 5. Consider the parity biquandle X with elements $\{1, 2, 3\}$ and operation matrix

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 3 & 3 & 1 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 3 & 1 & 1 & 3 & 1 \\ \hline 1 & 3 & 1 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 1 & 1 & 1 \end{array} \right].$$

Our python searches reveal that X has strongly compatible parity enhanced cocycles over $A = \mathbb{Z}_5$, including

$$\phi = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 2 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array} \right].$$

Then the virtual trefoil knot 2.1 has three X -colorings with Boltzmann weights as depicted:



yielding a parity-enhanced biquandle cocycle invariant value of $\Phi_X^\phi(2.1) = 2 + v$, distinguishing it from the unknot with $\Phi_X^\phi(0.1) = 3u^0 = 3$. On the other hand, the corresponding non-parity biquandle cocycle invariant (treating all crossings as even) has value $3u^0 = 3$ for both 2.1 and the unknot.

For our next example, we chose a four-element biquandle and strongly compatible parity enhanced cocycle over $A = \mathbb{Z}_3$ and computed the invariant for all prime virtual knots with up to four classical crossings as listed in the knot atlas [2].

Example 6. Let X be the parity biquandle with operation matrix

$$\left[\begin{array}{cccc|cccc} 3 & 4 & 2 & 1 & 3 & 4 & 2 & 1 \\ 1 & 2 & 4 & 3 & 1 & 2 & 4 & 3 \\ 4 & 3 & 1 & 2 & 4 & 3 & 1 & 2 \\ 2 & 1 & 3 & 4 & 2 & 1 & 3 & 4 \\ \hline 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 \\ 2 & 4 & 2 & 4 & 2 & 4 & 2 & 4 \\ 3 & 1 & 3 & 1 & 3 & 1 & 3 & 1 \\ 4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 \end{array} \right]$$

and ϕ the strongly compatible parity enhanced cocycle over \mathbb{Z}_3 with matrix

$$\left[\begin{array}{cccc|cccc} 0 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 0 & 1 & 1 & 1 & 1 \end{array} \right].$$

Then our `python` computations reveal values of the two-variable parity cocycle invariant $\Phi_X^{\phi,s}$ for the prime virtual knots with up to 4 classical crossings as listed in the table (numbered as in the knot atlas [2]). The double lines divide the table by parity biquandle counting invariant value $\Phi_X^{\mathbb{Z}_3}$ and the single lines divide the

table by single-variable parity cocycle enhancement value Φ_X^ϕ .

$\Phi_X^\phi(K)$	K
4	3.1, 3.5, 3.6, 3.7, 4.2, 4.6, 4.8, 4.12, 4.13, 4.17, 4.19, 4.26, 4.46, 4.47, 4.51, 4.55, 4.56, 4.75, 4.76, 4.77, 4.86, 4.93, 4.96, 4.97, 4.99, 4.102, 4.103, 4.105, 4.106, 4.108
$4u^2v$	4.29, 4.37, 4.61, 4.69
$4u$	4.36, 4.68
$4v$	3.2, 3.3, 3.4, 4.4, 4.5, 4.11, 4.18, 4.27, 4.44, 4.45, 4.49, 4.54, 4.74, 4.81, 4.82, 4.83, 4.87, 4.92, 4.94, 4.95, 4.101
$4u^2$	4.31, 4.41, 4.57, 4.65, 4.70
$4uv$	4.34, 4.40, 4.60, 4.64
$4v^2$	2.1, 4.1, 4.3, 4.7, 4.25, 4.28, 4.43, 4.53, 4.73, 4.80, 4.84, 4.88, 4.91, 4.100, 4.104
8	4.10, 4.16, 4.21, 4.23, 4.24, 4.50, 4.79
$8v$	4.9, 4.14, 4.15, 4.20, 4.22, 4.48, 4.52, 4.78
$4u^2 + 4u$	4.32, 4.35, 4.42, 4.58, 4.59, 4.66, 4.67, 4.71, 4.72
$4u^2v + 4uv$	4.30, 4.33, 4.38, 4.39, 4.62, 4.63
16	4.90, 4.98
$4u^2 + 12$	4.89
$4u^2 + 4u + 8$	4.107
$8u^2 + 4u + 4$	4.85

Example 7. For any parity biquandle X , the maps $\phi^0(x, y) = 0$ and $\phi^1(x, y) = 1$ for all $x, y \in X$ define a strongly compatible parity-enhanced cocycle. The resulting invariant $\Phi_X^{\phi, s}(X)$ has value

$$\Phi_X^{\phi, s}(K) = \Phi_X^{\mathbb{Z}}(K)v^{OW(K)}$$

where

$$OW(K) = \sum_{c \text{ odd crossing}} \epsilon(c)$$

is the *odd writhe* of K , the sum of crossing signs at odd crossings. In particular, if $\Phi_X^\phi(K) \neq \Phi_X^{\mathbb{Z}}(K)$ then K must be non-classical. Moreover, this example shows that cohomologous cocycles ϕ^0 and ψ^0 need not define the same parity-enhanced invariant, unlike the traditional case.

5 Questions

In what situations are two virtual knots distinguished by parity biquandle invariants but not by the corresponding biquandle invariants? In other words, what conditions are sufficient for the parity biquandle invariants to be stronger than their non-parity counterparts?

We would like to express the parity Boltzmann weight invariant in the language of cohomology. So far we don't have a satisfactory sense of "parity" for elements of C_3 , representing knotted surfaces, which leads to a useful boundary map. Specifically we would like a boundary map which yields 2-crossings of the correct parities. We also wonder about possible parity in C_1 .

A related question is what the relationship is between ϕ^0 and ϕ^1 in general? As we've seen, the set of ϕ^1 strongly compatible with a given ϕ^0 forms an A -module; what is the relationship of these modules with $C^2(X; A)$? Is there some deeper homology theory, perhaps with some "parity grading", from which the compatibility conditions emerge naturally?

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